Recall -- form factor:

$$F(q^2) \equiv \int e^{i\vec{q}\cdot\vec{r}} \rho(r)\, d^3r$$

inverse Fourier transform:

$$\rho(r) \equiv \frac{1}{(2\pi)^3} \int e^{-i\vec{q}\cdot\vec{r}} F(q^2)\, d^3q$$

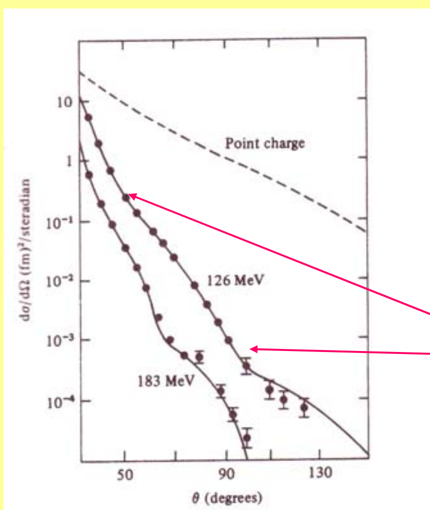
In principle, one could measure the form factor, and numerically integrate to invert the Fourier transform and find ρ(r).

However, this doesn't work in practice, because the integral has to be done over a complete range of q from 0 to ∞, and no experiment can ever span an infinite range of momentum transfer!

(It is bad enough trying to acquire data at large momentum transfer because the basic cross-section drops like q^{-4} → the rate of scattered particles into a detector gets too small – see lecture 4)

What to do? ...

Experimental data are fitted to a functional form for $F(q^2)$; parameters extracted from the fit are used to invert the transform and deduce ρ(r)....



Example: elastic electron scattering from gold (A=197, Z = 79). Best fit to data is given by solid curves.

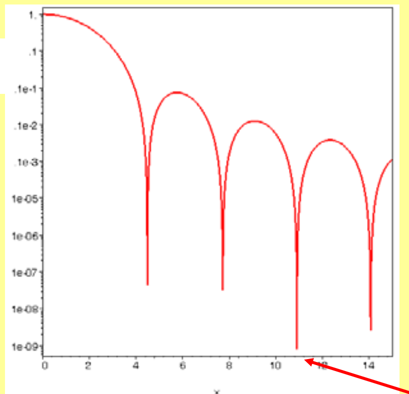
(Ref: R. Hofstadter, *Electron Scattering & Nuclear Structure*, 1963)

Discontinuities are evidence of diffraction-like behavior, characteristic of a Fourier transform, but the edges are fuzzy!

Illustration of "Diffraction" behavior in F(q^2): Uniform sphere
3

$$F(q^2) \equiv \int e^{i\vec{q}\cdot\vec{r}}\rho(r)\,d^3r = \int_0^{2\pi} d\phi \int_0^{\infty} r^2\rho(r)dr \int_0^{\pi} \sin\theta\, e^{iqr\cos\theta}\, d\theta$$

$\left(F(q^2)\right)^2$



Density: $\rho = \rho_o,\ r < R;\quad \rho = 0,\ r > R$

$$F(q^2) = \frac{3(\sin x - x\cos x)}{x^3}$$

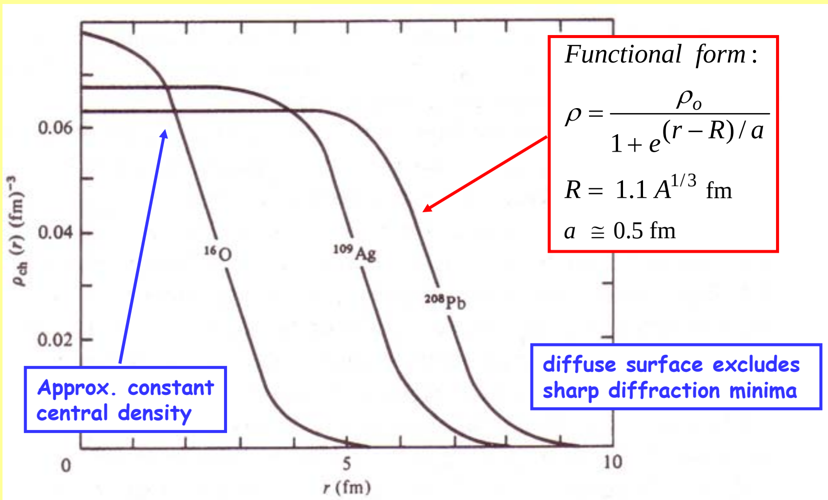
$\{ x = qR \}$

Important scaling property – in dimensionless variable (qR)

Zeroes never quite get to zero on a log scale!

In contrast, the **minima are not as sharp for nuclei**...

Nuclear charge distributions from experiment:
4



Functional form:

$$\rho = \frac{\rho_o}{1 + e^{(r-R)/a}}$$

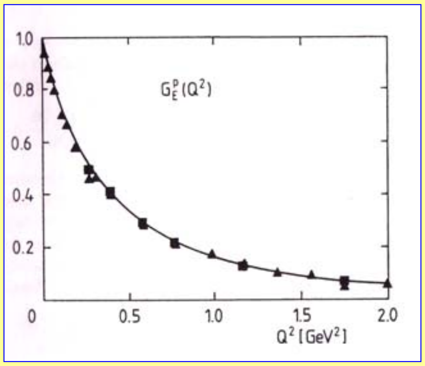
$R = 1.1\,A^{1/3}$ fm

$a \cong 0.5$ fm

Fig. 4.3 The electric charge density of three nuclei as fitted by $\rho_{ch}(r) = \rho^0_{ch}/[1 + \exp((r-R)/a)]$. The parameters are taken from the compilation in Barrett, R. C. & Jackson, D. F. (1977), *Nuclear Sizes and Structure*, Oxford: Clarendon Press.

Contrast to the proton (recall lecture 4): 5

Proton Form factor data:



$$G_E^p(Q^2) = \left(1 + \frac{Q^2}{0.71\,\mathrm{GeV}^2}\right)^{-2}$$

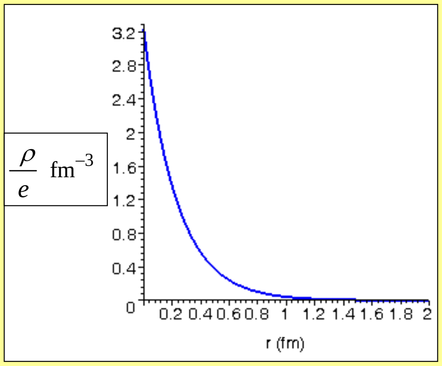
Fourier transform of charge density →

Electric charge distribution:

$$\rho(r) = e\rho_o \exp(-M\,r)$$

$$M = 4.33 \;\mathrm{fm}^{-1}$$

$$\left\langle r^2 \right\rangle^{1/2} = \frac{\sqrt{12}}{M} = 0.80 \;\mathrm{fm}$$



Understanding the form factor 6

$$F(q^2) \equiv \int e^{i\vec{q}\cdot\vec{r}}\rho(r)\,d^3r$$

Why is this a function of q^2 and not just q?

Famous and important result: the "**Form Factor Expansion**", derived as follows:

1) $$F(q^2) = \int\left[1 + i\vec{q}\cdot\vec{r} - (\vec{q}\cdot\vec{r})^2/2 + \ldots\right]\rho(r)\,d^3r$$
$$= \int\left[1 + iqr\cos\theta - (qr\cos\theta)^2/2 + \ldots\right]\rho(r)\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

2) $$\int\rho(r)\,d^3r = 1 \quad (\text{normalization}); \qquad \int_0^\pi \cos\theta\sin\theta\,d\theta = 0\ldots$$

$$\Rightarrow \quad F(q^2) = 1 - \frac{q^2}{2}\int r^2\cos^2\theta\,\rho(r)\,d^3r$$

continued... 7

but note the definition of the **mean square charge radius**: $\left\langle r^2 \right\rangle \equiv \int r^2 \rho(r)\, d^3 r$

The first nontrivial term in the expansion is proportional to $\langle r^2 \rangle$ (details for homework!)

Result: $\Rightarrow$

$$F(q^2) = 1 - \frac{q^2}{6}\left\langle r^2 \right\rangle + \;....$$

(F&H eq. 6.28)

Think carefully:

1. This expansion only applies at small momentum transfer; it does not replace the exact result obtained by integration over the charge density explicitly.
2. The result is universal, independent of any details of $\rho(r)$ except for the mean square charge radius.
3. We can use it to assess when "pointlike" behavior should be observed, and what q^2 range is necessary to observe details of the structure of the scattering object,

   e.g. for $(1 - F(q^2)) > 0.1$, we require $q^2 > 0.6/\langle r^2 \rangle$

continued.... 8

$$F(q^2) = 1 - \frac{q^2}{6}\left\langle r^2 \right\rangle + \;....$$ $\Rightarrow$

- To see the structure of a nucleus like ^{208}Pb, $\langle r^2 \rangle = (1.2\, A^{1/3})^2 = 50$ fm^2,
  we need $q^2 \geq\approx 0.01$ fm^{-2} ≈ 400 (MeV/c)^2
- For a proton, $\langle r^2 \rangle = 0.64$ fm^2, we need $q^2 \geq\approx 1$ fm$^{-2} \approx 40{,}000$ (MeV/c)^2 = 0.04 (GeV/c)^2
- Since we know that q^2 is limited by the incident beam momentum, we need a much higher energy accelerator facility to map out the structure of the proton than to study heavy nuclei

  (e.g. SLAC experiment, data table shown in lecture 4, used beam energies in the range 5 – 21 GeV and scattering angles in the range 20° – 30° to map out the proton form factor at "large momentum transfer"...)

Next: How does our cross-section formula change if proper relativistic quantum mechanics is used, so that we can correctly apply it to the proton?

9

Relativistic Formulation: spin ½ beam and target

1. Differential cross-section for scattering of spin-½ electrons from a pointlike (spin 0) target is given by the "**Mott cross-section**", which is our result multiplied by $\cos^2(\theta/2)$ and a "recoil factor" E'/E_o

$$\left.\frac{d\sigma}{d\Omega}\right|_{Mott} = \frac{(\hbar c\alpha)^2}{4E_o^2\sin^4(\theta/2)}\left(\frac{E'}{E_o}\right)\cos^2(\theta/2)$$

$$= \frac{(\hbar c\alpha)^2\cos^2(\theta/2)}{4E_o^2\sin^4(\theta/2)\left[1+2\frac{E_o}{M}\sin^2(\theta/2)\right]}$$

fine structure constant, α:

$$\alpha = \frac{e^2}{4\pi\varepsilon_o\hbar c} \cong \frac{1}{137}$$

2. In terms of the **4-momentum transfer, Q^2**, the result for an **extended spin ½ target**, with both electric charge and magnetization distributions, is formulated in terms of **electric and magnetic form factors**:

$$G_E(Q^2) \quad and \quad G_M(Q^2) \; ...$$

NB. Notation is from kinematics, lecture 5

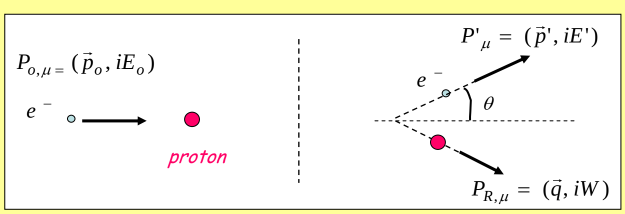
10

Form factors of the proton: "Rosenbluth formula": (F&H sec. 6.7)

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{Mott}\left\{\left(\frac{G_E^2+\tau\, G_M^2}{1+\tau}\right) + 2\tau\, G_M^2\,\tan^2(\theta/2)\right\}, \quad with \;\; \tau \equiv \frac{Q^2}{4M^2}$$

Recall 4-momentum from lecture 5:

$$p_\mu = (\vec{p},\; iE), \;\; \mu = 1...4\,; \qquad \sum_\mu p_\mu^2 = p^2 - E^2 = -m^2$$



$$Q \equiv (P_o - P') \quad \Rightarrow \quad Q^2 = 2p_o p'(1-\cos\theta) = 4\,E_o E'\sin^2(\theta/2)$$

invariant!!!

What does this mean?

11

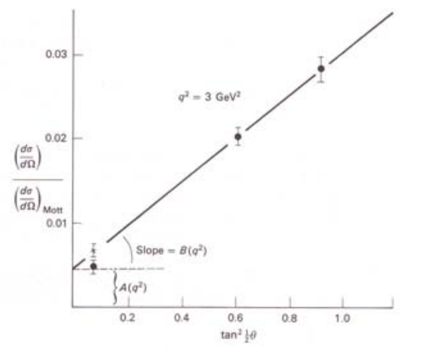
$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{Mott} \left\{ \left(\frac{G_E^2 + \tau\, G_M^2}{1+\tau} \right) + 2\tau\, G_M^2 \tan^2(\theta/2) \right\}, \quad with\ \tau \equiv \frac{Q^2}{4M^2}$$

- both electric and magnetic form factors contribute to the scattering
- to disentangle the two contributions, one has to compare measurements at the **same Q^2** but **different scattering angles** θ

"Rosenbluth separation method":

$$\frac{\frac{d\sigma}{d\Omega}}{\left(\frac{d\sigma}{d\Omega}\right)_{Mott}} = \left\{ A(Q^2) + B(Q^2)\tan^2\frac{\theta}{2} \right\}$$

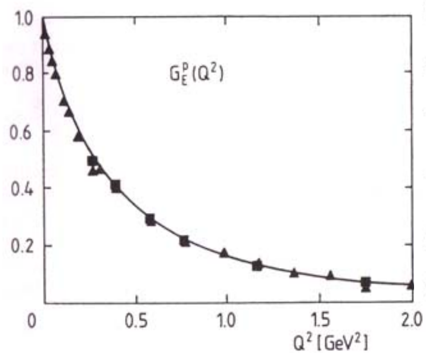
Example: e-p scattering ($q^2 = Q^2$ here)



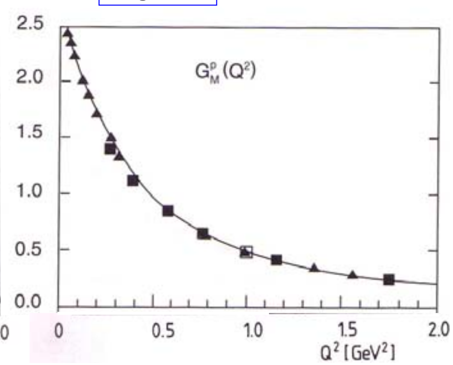
Electromagnetic form factors of the proton:

12

Electric:



Magnetic:



Same shape!!! scaling relation:

$$G_E^p(Q^2) = G_M^p(Q^2)/\mu_N = \frac{1}{(1+Q^2/0.71)^2}$$

Interpretation of the form factors: 13

1. **At $Q^2 = 0$, we have:**

$G_E(0) = 1$ (pointlike limit, $G_E(0)$ = normalized electric charge)

$G_M(0) = \mu_p$ (" " , $G_M(0)$ = magnetic moment)

2. Evaluated in the center-of-mass reference frame, with initial state 3-momenta which exactly reverse after the collision:

$$\vec{p}_{i,cm} = \tfrac{1}{2}\,\vec{q} \quad (electron) \qquad and \qquad \vec{p}_{p,cm} = -\tfrac{1}{2}\,\vec{q} \quad (proton)$$

The 4-momentum transfer is then: $Q_\mu = (\vec{q}, 0) \quad and \quad Q^2 = q^2 \; ...$

and the charge density is:

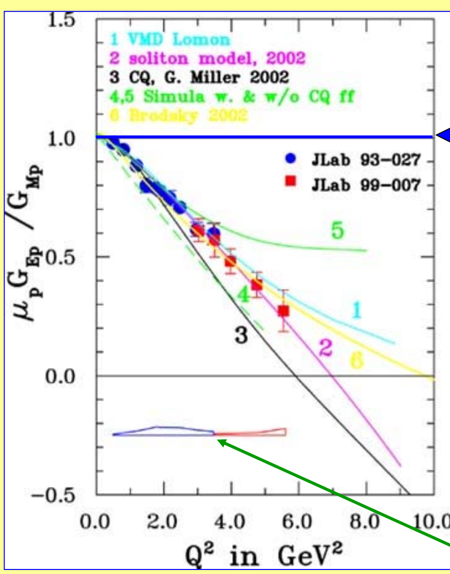
$$\rho(\vec{r}) = \frac{1}{(2\pi)^3} \int \frac{M}{E}\, e^{-i\vec{q}\cdot\vec{r}}\, G_E(q^2)$$

(M/E ratio is for the proton)

The magnetization density is obtained in a similar manner from G_M

Recent discovery – Jefferson Lab data *(see US Long Range Plan, lecture 2)* 14

Jefferson Lab Research Highlights: http://www.jlab.org/highlights/nuclear/Nuclear.html



(new, high precision data taken using an alternative technique, complementary to the Rosenbluth method.)

Old scaling law: $\mu G_E/G_M$=1

(Various new theoretical predictions, colored lines 1 .. 5)

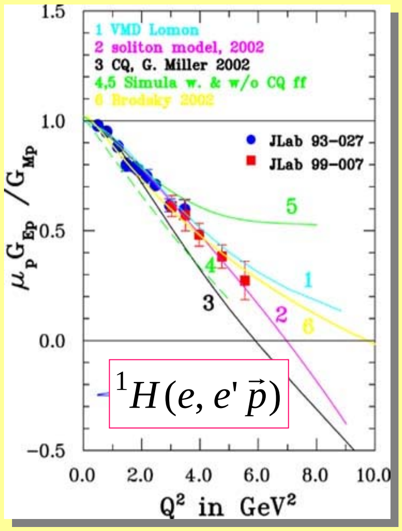
"For more than 20 years it has been assumed, based on the available data, that the charge and magnetization distributions in the proton were proportional to one another (corresponding to µGe/Gm=1). New data from Jefferson Lab shows this is not true, and is leading to a re-examination of the dynamics governing the proton's quark wavefunctions."

M. K. Jones, et al., Phys. Rev. Lett. 84, 1398 (2000),
O. Gayou, et. al., Phys. Rev. Lett. 88, 092301 (2002)

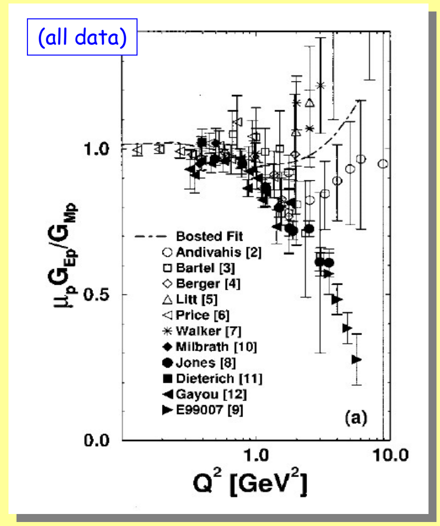
systematic error limits

Proton G_E versus G_M scaling controversy

JLab "polarization transfer" data, measures ratio G_E/G_M directly:



Open circles from Rosenbluth separation, G_E and G_M determined separately → ratio.



The proton: where are we?

- charge and magnetization distributions are very similar
- both the form factors appear to follow a "dipole" pattern, e.g.

$$G_E^p(Q^2) = \frac{1}{(1+Q^2/0.71\ \text{GeV}^2)^2}$$

- new measurements from Jefferson Lab show that the charge and magnetization distributions are increasingly different at higher Q^2 for reasons that are not yet fully understood
- Next: the proton has excited states!

Electric charge distribution:

$$\rho(r) = e\rho_o \exp(-M\, r)$$

$$M = 4.33\ \text{fm}^{-1}$$

$$\langle r^2 \rangle^{1/2} = \frac{\sqrt{12}}{M} = 0.80\ \text{fm}$$